Binomial Series

Fact (Geometric Series) — $(1-x)^{-1} = 1 + x + x^2 + \cdots$ if $|x| < 1$

Example

Given the fact about geometric series what is the series expansion of:

(a) $(1-x)^{-2}$

(b) $(1-x)^{\frac{1}{2}}$

(a)

$$\begin{aligned}(1-x)^{-2} &= (1-x)^{-1} \cdot (1-x)^{-1} \\ &= (1+x+x^2+\cdots)(1+x+x^2+\cdots) \\ &= 1 + x(1+1) + x^2(1+1+1) + x^3(1+1+1+1) + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots\end{aligned}$$

(b) Suppose $(1-x)^{\frac{1}{2}} = a_0 + a_1 x + a_2 x^2 + \cdots$ *then*

$$\begin{aligned}1 - x &= (1-x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}} \\ &= (a_0 + a_1 x + a_2 x^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0^2 + (2a_1 a_0)x + (2a_0 a_2 + a_1^2)x^2 + \cdots \\ \Rightarrow \qquad a_0^2 &= 1 \\ 2a_1 a_0 &= -1 \\ 2a_0 a_2 + a_1^2 &= 0 \\ \Rightarrow \qquad a_0 &= 1 \\ a_1 &= -\tfrac{1}{2} \\ a_2 &= -\tfrac{1}{8}\end{aligned}$$

Example

Compute

- $(1-x)^{-2}$
- $(1-x)^{-3}$
- $(1-x)^{-4}$
- $(1-x)^{-5}$

Example

Compute $\frac{1}{1-x} \cdot \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)$

Fact (Generalised Binomial Coefficients) —

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \qquad \text{if } \alpha \in \mathbb{C}$$

Example

Prove that if $n \in \mathbb{Z}$

$$(1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r \qquad \text{if } |x| < 1$$

Lemma

Prove that $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$

$$\begin{aligned}
\binom{-n}{k} &= \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} \\
&= (-1)^k \frac{n(n+1)\cdots(n+k-1)}{k!} \\
&= (-1)^k \frac{(n+k-1)!}{(n-1)!k!} \\
&= (-1)^k \binom{n+k-1}{k} = (-1)^k \binom{n+k-1}{n-1}
\end{aligned}$$

Fact (Generalised Binomial Theorem) — If $\alpha \in \mathbb{C}$

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r = \sum_{r=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} x^r \qquad \text{if } |x|<1$$

Example

Expand $\sqrt[4]{1+2x}$ as an infinite convergent Binomial series, up to and including the term in x^4. (State the range of values of x for which it is valid).

$$\begin{aligned}
\sqrt[4]{1+2x} &= (1+2x)^{\frac{1}{4}} \\
&= 1 + \frac{\frac{1}{4}}{1!}(2x)^1 + \frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)}{2!}(2x)^2 + \frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)\cdot\left(\frac{1}{4}-2\right)}{3!}(2x)^3 + \frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)\cdot\left(\frac{1}{4}-2\right)\cdot\left(\frac{1}{4}-3\right)}{4!}(2x)^4 + \cdots \\
&= 1 + \tfrac{1}{2}x + \frac{\frac{1}{4}\cdot\left(-\frac{3}{4}\right)}{2}\cdot 4x^2 + \frac{\frac{1}{4}\cdot\left(-\frac{3}{4}\right)\cdot\left(-\frac{7}{4}\right)}{6}8x^3 + \frac{\frac{1}{4}\cdot\left(-\frac{3}{4}\right)\cdot\left(-\frac{7}{4}\right)\cdot\left(-\frac{11}{4}\right)}{24}16x^4 + \cdots \\
&= 1 + \tfrac{1}{2}x - \tfrac{3}{8}x^2 + \tfrac{7}{16}x^3 - \tfrac{77}{128}x^4 + \cdots
\end{aligned}$$

if $|2x|<1 \Leftrightarrow |x|<\frac{1}{2}$

Example

Expand $\sqrt[3]{8+24x}$ as an infinite convergent Binomial series, up to and including the term in x^3. (State the range of values of x for which it is valid).

$$\begin{aligned}
\sqrt[3]{8+24x} &= (8+24x)^{\frac{1}{3}} \\
&= (8(1+3x))^{\frac{1}{3}} \\
&= 2(1+3x)^{\frac{1}{3}} \\
&= 2\left(1 + \frac{\frac{1}{3}}{1!}(3x)^1 + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)}{2!}(3x)^2 + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot\left(-\frac{5}{3}\right)}{3!}(3x)^3 + \cdots\right) \\
&= 2\left(1 + x - x^2 + \tfrac{5}{3}x^3 + \cdots\right) \\
&= 2 + 2x - 2x^2 + \tfrac{10}{3}x^3 + \cdots \qquad \textit{if } |x|<\tfrac{1}{3}
\end{aligned}$$

Example

By considering $(1-x)^{-1}$ and differentiating, find

$$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \sum_{r=1}^{\infty} r^2 x^{r-1}$$

$$\begin{aligned}
& & (1-x)^{-1} &= 1 + x + x^2 + \cdots & &= \sum_{r=0}^{\infty} x^r \\
&\frac{d}{dx} & (1-x)^{-2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots & &= \sum_{r=0}^{\infty} r x^{r-1} \\
&x\cdot & x(1-x)^{-2} &= x + 2x^2 + 3x^3 + 4x^4 + \cdots & &= \sum_{r=0}^{\infty} r x^{r} \\
&\frac{d}{dx} & (1-x)^{-2} + 2x(1-x)^{-3} &= 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots & &= \sum_{r=0}^{\infty} r^2 x^{r-1} \\
&\Rightarrow & \sum_{r=0}^{\infty} r^2 x^{r-1} &= \frac{1+x}{(1-x)^3}
\end{aligned}$$

Example

Hence or otherwise, find $1^2 + 2^2 + 3^2 + \cdots + n^2$

Notice that $1^2 + (1^2+2^2)x + (1^2+2^2+3^2)x^2 + \cdots = \frac{1}{1-x} \cdot \frac{1+x}{(1-x)^3} = (1+x)(1-x)^{-4}$*, so we need to look at the coefficient of* x^{n-1}

We obtain this from the coefficient of x^{n-1} *in* $(1-x)^{-4}$ *and the coefficient of* x^{n-2}*, which are* $\binom{-4}{n-1}$ *and* $\binom{-4}{n-2}$*, ie*

$$\begin{aligned}
S &= (-1)^{n-1}\binom{-4}{n-1} + (-1)^{n-2}\binom{-4}{n-2} \\
&= \binom{n-1+4-1}{n-1} + \binom{n-2+4-1}{n-2} \\
&= \binom{n+2}{n-1} + \binom{n+1}{n-2} \\
&= \binom{n+2}{3} + \binom{n+1}{3}
\end{aligned}$$